TECHNICAL REPORT

R-160

CONSIDERATIONS OF THE MOTION OF A SMALL BODY

IN THE VICINITY OF THE STABLE LIBRATION POINTS

OF THE EARTH-MOON SYSTEM

By William H. Michael, Jr.

Langley Research Center
Langley Station, Hampton, Va.

ADVANCE COPY

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL REPORT R-160

CONSIDERATIONS OF THE MOTION OF A SMALL BODY

IN THE VICINITY OF THE STABLE LIBRATION POINTS

OF THE EARTH-MOON SYSTEM

By William H. Michael, Jr.

ABSTRACT

On the basis of a first-order analysis of the equations of motion of the restricted three-body problem, an explicit solution is obtained for the position and velocity components of the motion as a function of the initial conditions. The analytical results are compared with results obtained by numerical integration.

TABLE OF CONTENTS

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL REPORT R-160

CONSIDERATIONS OF THE MOTION OF A SMALL BODY

IN THE VICINITY OF THE STABLE LIBRATION POINTS

OF THE EARTH-MOON SYSTEM[1]

By William H. Michael, Jr.

SUMMARY

An analytical investigation is performed to determine the properties of the motion of a small body in the vicinity of the stable libration points of the Earth-Moon system. On the basis of a first-order analysis of the equations of motion of the restricted three-body problem, an explicit solution is obtained for the position and velocity components of the motion as a function of the initial conditions. The results of the explicit solution lead to a determination of the properties of the motion as a function of the initial conditions. The properties so determined include the position in orbit and the velocity components as a function of time, the properties of the component elliptical motions which comprise the resultant motion, the existence of periodic elliptical orbits through proper choice of initial conditions, and the limits of the motion. The analytical results are compared with results obtained by numerical integration of the equations of motion on a high-speed digital computer; the comparisons indicate good agreement even for rather large excursions from the stable libration points.

Suggestions are given for possible scientific experiments which might be performed with use of artificial satellites established in orbit about the stable libration points. With respect to establishment of such satellites, consideration is given to trajectory optimization and a dispersion analysis.

With respect to a recent report by K. Kordylewski of the Cracow Observatory of observations of cloud-like natural satellites in the vicinity of the trailing stable libration point of the Earth-Moon system, several possible mechanisms for particle accumulation are considered.

---

[1]This material represents a modification of a thesis presented to the College of William and Mary in partial fulfillment of the requirements for the degree of Master of Arts in Physics, May 1962.

INTRODUCTION

The restricted three-body problem of celestial mechanics treats the relative motion of a body of negligible mass subject to the gravitational attraction of two finite point masses which are assumed to revolve in circles about their common center of mass with uniform angular velocity. This problem has been treated by a number of investigators, and the classical results are summarized in textbooks such as references 1 and 2. Although the exact conditions corresponding to the given assumptions are not found in nature, a few natural configurations are sufficiently close to the assumed conditions to make this problem of great practical interest. The motion of a small body relative to the Sun and Jupiter, relative to the Earth and Moon, and to a lesser extent, the motion of the Moon relative to the Earth and Sun are specific examples. Treatment of such problems through the use of the restricted three-body problem, even in view of the approximations involved, is often sufficient to demonstrate the fundamental, and in some cases the particular, properties of the motion.

Lagrange first determined the particular solutions of the restricted three-body problem. (See ref. 1.) There are five particular positions relative to the two finite masses having properties such that a small body placed at these positions with the proper initial conditions will remain there indefinitely unless acted upon by forces exterior to the system under consideration. These positions are called libration points and all five are located in the plane of motion of the two finite bodies. Three are located along the line joining the two finite masses and two are so located that the two finite masses and the libration points form equilateral triangles. The three colinear libration points are unstable, in the sense that an infinitesimal body initially displaced a small amount from one of the points would generally depart to a comparatively large distance. On the other hand, the equilateral triangle libration points are stable so long as the ratio of the secondary mass to the primary mass is sufficiently small. Discussions of the libration points of the restricted three-body problem are contained in several texts on celestial mechanics, particularly in those of Moulton (ref. 1) and Plummer (ref. 2).

The mass ratios of the Earth-Moon system and of the Sun-Jupiter system satisfy the criterion for stable motion of a small body relative to the equilateral triangle libration points of these systems. With respect to the Sun-Jupiter system, nature has provided a remarkable example of this stability in the existence of a group of asteroids situated in the neighborhood of the stable libration points and in motion about these points. The first of these asteroids was discovered in 1906 and thirteen such asteroids have been discovered thus far, eight of which are in motion about the libration point which precedes Jupiter and five about that which follows Jupiter.

With respect to the Earth-Moon system, a very recent report of the discovery of two cloud-like satellites in the vicinity of the trailing stable libration point has been made by K. Kordylewski of the Cracow Observatory. Some details of this report are given in references 3 and 4. This is the only known report of sightings of objects orbiting about the libration points of the Earth-Moon system. Several observatories and the Smithsonian Astrophysical Observatory Baker-Nunn

tracking camera network are making plans to conduct observations at appropriate times. These recent developments provide an impetus for theoretical studies of the motion of a small body in the vicinity of the stable libration points of the Earth-Moon system.

Independent of the existence of natural satellites at the libration points, the establishment of artificial satellites in these regions appears possible. The problem is in several aspects similar to that of establishing a satellite about the Moon. In both problems sufficient velocity must be supplied to project a vehicle to the desired position relative to the libration point or to the Moon; then additional thrust must be applied to reduce the relative velocity of the vehicle to a value below that of the effective escape velocity in order to effect capture. The velocities and accuracies required to perform such an operation in the case of a lunar satellite have been determined and have been discussed in some detail in the recent literature. (See, for instance, ref. 5.) It is of some interest to determine accuracy requirements for the establishment of a satellite about the stable libration points, in conjunction with a discussion of those properties of the motion which are determinable.

The question arises as to the practical application of the establishment of an artificial satellite in the vicinity of the stable libration points of the Earth-Moon system. In reference 6 it is suggested that observation of the motion of such a satellite, particularly the determination of the periodicities of the motion, could result in a more accurate determination of the Earth-Moon mass ratio. The basis of this suggestion is that, in the absence of perturbations external to the Earth-Moon system, the periodicities are a function only of the mass ratio. It has also been suggested that satellites at the libration points could be used as way stations for deep space exploration, for storage of fuel or other supplies, or for Earth reconnaissance.

The suggestion is offered that a satellite about the stable libration points might be used as a solar or cosmic radiation monitor. Such a monitor could supply significant information, in conjunction with Earth-based or Earth satellite measurements, regarding the velocity, directionality, and intensity of the various incident radiation components in the vicinity of the Earth-Moon system. Such information, and other scientific instrumentation, could provide additional knowledge on the nature of solar flares, magnetic and electric fields in space, dust and micrometeoroid population, and the Earth's magnetic field.

The primary consideration of the present study is the determination of the properties of the motion of a small body in the near vicinity of the stable libration points of the Earth-Moon system, on the basis of a first-order analysis of the equations of motion of a three-body system. As a consequence of determining properties of the motion, some considerations concerning the establishment of artificial satellites about the libration points can be presented. With respect to possible accumulation of particles in the vicinity of the stable libration points which might be observable from the Earth, several possible mechanisms for such accumulation are considered.

C	a constant
C_1 to C_5	Jacobi's constant for surfaces of zero relative velocity, see figure 2
e	eccentricity of elliptical orbit
K_n, L_n	coefficients used in solution of equations of motion
M_n, N_n	constants depending on initial conditions, defined in equations (23)
P	sidereal period of Moon
r_1	radius from Earth to small body
r_2	radius from Moon to small body
U	function defined in equation (2)
V	relative velocity of particle with respect to libration point
W	function defined in equation (8)
X, Y, Z	coordinate axes, defined in figure 1
X', Y', Z'	transformed coordinate axes, defined in figure 3
x, y, z	denotes position of small body in X, Y, Z system
x', y', z'	denotes position of small body in X', Y', Z' system
x_1, x_2	distance from Earth or Moon, respectively, to center of mass of Earth-Moon system
α, β	constants defined in equations (10)
γ_n	constants defined in equation (18)
θ	angle defined in transformation of coordinates
λ_n'	imaginary constants, defined in equations (15)
λ_n	frequencies defined in equations (20)
μ	ratio of mass of Moon to total mass of Earth and Moon, taken as $\frac{1}{82.45}$

4

ξ,η transformed coordinate axes, defined in figure 3; also denotes position of small body in this system

Subscript:

o   denotes initial conditions

PRELIMINARY CONSIDERATIONS

Equations of Motion for the Restricted Three-Body Problem

A simplification of the equations of motion of an infinitesimal body in motion relative to and under the gravitational attraction of two finite point masses can be obtained when the equations are written relative to a rotating coordinate system. If the two finite point masses are assumed to revolve in circles about their common center of mass with uniform angular velocity, the coordinate system can be chosen so that the equations of motion of the infinitesimal body do not involve an explicit dependence on the time. With the origin of the coordinate system taken at the center of mass, the X-axis along the line joining the two finite masses and positive in the direction of the smaller mass, the Y-axis normal to the X-axis and positive in the direction of motion of the smaller mass, and the Z-axis such as to complete a right-hand system (see fig. 1), the equations of motion of the infinitesimal body are

$$
\left.
\begin{aligned}
\frac{d^2x}{dt^2} - 2\,\frac{dy}{dt} &= x - (1-\mu)\frac{(x-x_1)}{r_1^3} - \mu\,\frac{(x-x_2)}{r_2^3} \\[2ex]
\frac{d^2y}{dt^2} + 2\,\frac{dx}{dt} &= y - (1-\mu)\frac{y}{r_1^3} - \mu\,\frac{y}{r_2^3} \\[2ex]
\frac{d^2z}{dt^2} &= -(1-\mu)\frac{z}{r_1^3} - \mu\,\frac{z}{r_2^3}
\end{aligned}
\right\}
\tag{1}
$$

In equations (1) the constants x_1 and x_2 denote the position of the primary and secondary masses, respectively, with respect to the common center of mass. The quantities $(1-\mu)$ and μ denote, respectively, the primary and secondary mass ratios. The quantities r_1 and r_2 are the radii of the infinitesimal body from $(1-\mu)$ and μ, respectively, and are defined as

$$
r_1 = \sqrt{(x-x_1)^2 + y^2 + z^2}
$$
$$
r_2 = \sqrt{(x-x_2)^2 + y^2 + z^2}
$$
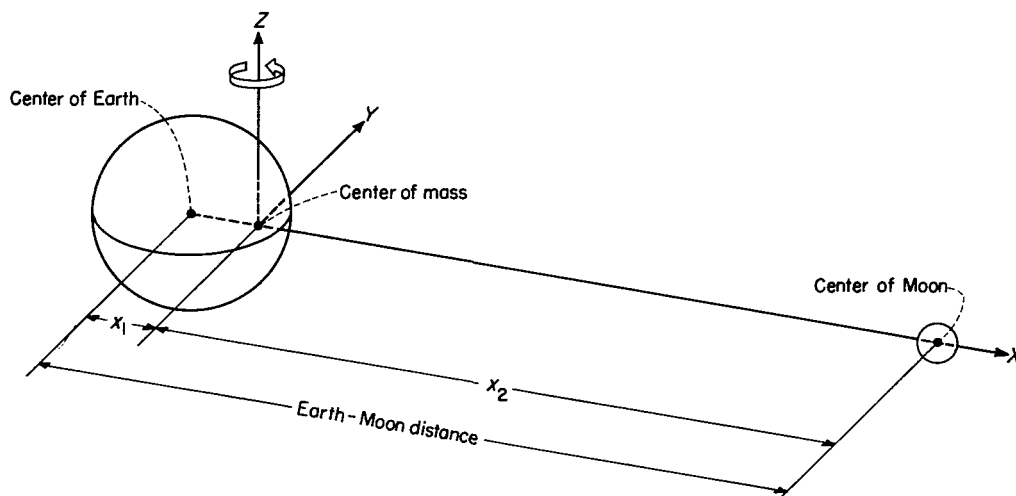
Figure 1.- Rotating coordinate system for three-body problem.

All factors in equations (1) are expressed in nondimensional form, with the unit of mass taken as the sum of the masses of the two finite bodies, the unit of distance taken as the distance between the finite bodies, and the unit of time (Period/2π) as the inverse of the mean angular motion of the two finite bodies.

The equations of motion of the restricted three-body problem, expressed in the form of equations (1), are derived in several textbooks on celestial mechanics, for example, in reference 1.

Jacobi's Integral and the Surfaces of Zero Relative Velocity

An integral of the equations of motion (eqs. (1)) was first obtained by Jacobi through definition of a function U, defined as

$$U = \frac{1}{2}\left(x^2 + y^2\right) + \frac{(1 - \mu)}{r_1} + \frac{\mu}{r_2} \tag{2}$$

Jacobi's integral, obtained by simple manipulation (ref. 1) of equations (1) and (2) is

$$V^2 = 2U - C \tag{3}$$

where C is a constant depending on the initial conditions.

6

Although Jacobi's integral is by no means a complete solution to the three-body problem, it does provide some useful information concerning the motion. As seen from equation (3), when the constant C is determined from the initial conditions, the limits of the motion and the velocity at any point within the limits of the motion can be obtained. Since the left side of equation (3) is a real quantity, the motion, for a given value of C, is real only for those points for which $2U$ is greater than or equal to C. Setting $V = 0$ allows calculation of the so-called surfaces of zero relative velocity which define the limits of the motion for a given value of C. With application to the lunar theory, G. W. Hill used this relation and these approximations to show that the Moon cannot recede beyond a certain limiting distance from the Earth. (See ref. 1.) In recent years a number of investigators have discussed at length the various implications of these surfaces with respect to projecting a vehicle from the Earth to the Moon.

Some of the surfaces of zero velocity for the Earth-Moon system are sketched, for the X-Y plane, in figure 2. With the value of the mass ratio adopted in this analysis, $\mu = \dfrac{1}{82.45} = 0.0121286$, the values of the constant C for the various curves are shown on figure 2.

Figure 2.- Sketch of surfaces of zero relative velocity for the Earth-Moon system.

The Libration Points

Figure 2 indicates that at certain points the curves of zero relative velocity either come together or vanish, as at point L_4 (and at a similarly situated reflected point L_5). These points, of which there are a total of five, all situated in the X-Y plane, represent critical points of the motion relative to the

7

rotating coordinate system. At these points the gradient of U (eq. (2)) is zero. The points therefore represent positions of equilibrium relative to the rotating coordinate system, so that a small body placed at any of these points with the proper initial conditions will remain there unless acted upon by a disturbance outside the considerations of the restricted three-body problem. The points L_1 to L_5 are the libration points of the Earth-Moon system, or the Lagrangian points, after the first investigator who determined their properties.

It can be shown that the three points along the X-axis are unstable. This can be visualized to some extent by consideration of the values of the constant C associated with these points. On the other hand, points L_4 and L_5 are stable points, in the sense that a small body given a small velocity relative to these points will continue to move in the proximity of the points. The points L_4 and L_5 are frequently referred to as the equilateral-triangle libration points. The criterion for stable motion, as derived in reference 1, is that $\mu < 0.0385$. Thus, if μ (the ratio of the mass of the secondary body to the total mass of the two finite bodies) is less than 0.0385, motion about the equilateral-triangle libration points is stable. The value of μ for the Earth-Moon system is about 0.0121, and for the Sun-Jupiter system, about 0.00095; therefore, in both these systems the motion about the equilateral-triangle libration points is stable.

Equations of Motion Relative to the Equilateral-Triangle

Libration Points

The equations of motion in the restricted three-body problem, obtained by combining equations (1) and (2) are

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= \frac{\partial U}{\partial x} \\[2mm]
\frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= \frac{\partial U}{\partial y} \\[2mm]
\frac{d^2z}{dt^2} &= \frac{\partial U}{\partial z}
\end{aligned}\right\} \tag{4}$$

The libration points constitute particular solutions to these equations. Let the coordinates of the small body relative to the equilateral-triangle libration points be designated x', y', and z' in a coordinate system with axes parallel to the original axes and centered at the libration points. (See fig. 3.) As shown in reference 1, equations (4) can be written with respect to the libration point L_4 by expansion of the right-hand sides in Taylor's series and
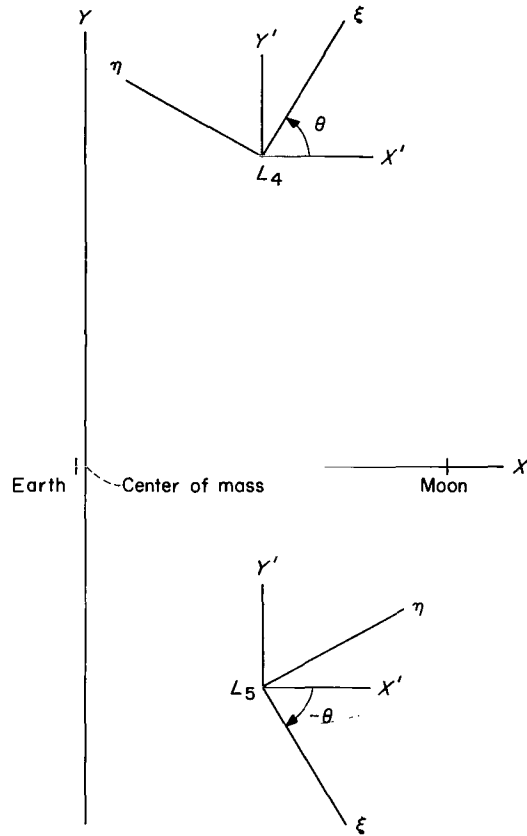
Figure 3.- Schematic diagram of coordinate systems.
The Z and Z' axes are positive upward from
plane of paper.

neglecting factors involving terms of the second degree and higher. The first-order equations of motion of the small body, relative to libration point L_4, obtained by this expansion are

$$\frac{d^2x'}{dt^2} - 2\frac{dy'}{dt^2} = \frac{3}{4}x' + \frac{3\sqrt{3}}{4}(1 - 2\mu)y'$$

$$\frac{d^2y'}{dt^2} + 2\frac{dx}{dt} = \frac{3\sqrt{3}}{4}(1 - 2\mu)x' + \frac{9}{4}y' \qquad (5)$$

$$\frac{d^2z'}{dt^2} = -z'$$

9

The equations of motion relative to point L_5 are the same as those given in equations (5), except for negative signs before the terms involving $\frac{3\sqrt{3}}{4}(1 - 2\mu)$ in equations (5).

The last of equations (5) can be immediately integrated. The results of the integration, in terms of the initial position and velocity components, designated z'_0 and $\left(\frac{dz'}{dt}\right)_0$, respectively, are

$$
\left.
\begin{aligned}
z' &= z'_0 \cos t + \left(\frac{dz'}{dt}\right)_0 \sin t \\[2ex]
\frac{dz'}{dt} &= -z'_0 \sin t + \left(\frac{dz'}{dt}\right)_0 \cos t
\end{aligned}
\right\}
\tag{6}
$$

Thus, the motion of the small body parallel to the z' axis, in the vicinity of the libration points, is periodic with frequency $2\pi/P$; that is, the period is the same as that of the revolution of the Moon about the Earth (the sidereal period of the Moon). Its position and velocity components at any time are easily calculated with given initial conditions.

ANALYSIS OF MOTION ABOUT STABLE LIBRATION POINTS

Transformation of Coordinates and a First Integral of the Problem

The first two of equations (5) admit a transformation of coordinate axes which simplifies the form of the equations. The transformation to the ξ, η axis system (fig. 3) is defined by

$$x' = \xi \cos \theta - \eta \sin \theta$$

$$y' = \xi \sin \theta + \eta \cos \theta$$

where θ is given by

$$\tan 2\theta = \frac{\pm \dfrac{3\sqrt{3}}{2}(1 - 2\mu)}{\dfrac{3}{4} - \dfrac{9}{4}}$$

With the value for μ adopted in the present analysis ($\mu = 0.0121286$), $\theta = \pm 60.306°$. Because of sign differences in applicable portions of equations (5)

10

for the points L_4 and L_5, the coordinate axes for the ξ,η system for point L_4 are obtained by a counterclockwise rotation through θ from the X',Y' coordinate axes and for the point L_5 by a clockwise rotation. (See fig. 3.)

After the numerical calculations for the transformation are carried out, the first-order equations of motion for a small body relative to the stable libration points become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - 2\frac{d\eta}{dt} = \alpha\xi = \frac{\partial W}{\partial \xi} \\[2mm] \frac{d^2\eta}{dt^2} + 2\frac{d\xi}{dt} = \beta\eta = \frac{\partial W}{\partial \eta} \end{aligned}\right\} \tag{7}$$

where

$$2W = 2.972795\xi^2 + 0.027205\eta^2 \tag{8}$$

Equations (7) and (8) are equally applicable to motion about point L_4 or L_5 subject to the orientation of the ξ,η coordinates as indicated in figure 3.

A first integral of equation (7) can be obtained in the same manner as the procedure for obtaining Jacobi's integral. If the first of equations (7) is multiplied by $2\frac{d\xi}{dt}$, the second by $2\frac{d\eta}{dt}$, and the resulting two equations are added, an equation in which the components are exact differentials is obtained. This equation is integrated to yield

$$V^2 = 2W + C \tag{9}$$

where W is defined in equation (8).

The constant C in equation (9) is easily determined from the velocity and the position components at the initial time. The first integral then gives the relation between velocity and the position components for any other position. The question of whether the small body can attain a particular position with given initial conditions is not answered by the first integral and this is an indication of the limited amount of information contained in it. This question is considered in detail in a subsequent section.

The first integral states that the velocity increases as the small body departs from the libration point. A simple example is a situation in which the initial position is at the libration point ($\xi_0 = 0$; $\eta_0 = 0$) and the initial velocity is V_0. In this case $C = V_0^2$ and the velocity at any point other than the libration point must be greater than V_0. The magnitude of the velocity is strongly dependent on the position components, particularly on the value of ξ.

There are no positions of zero relative velocity except at the libration points, and there is thus no possibility of discussing the surfaces of zero relative velocity as has been done in the case of discussions of the three-body problem on the basis of Jacobi's integral.

Equations (7), along with the definition of the coordinates in figure 3, indicate that the position-dependent terms alone lead to accelerations tending to carry a particle away from the libration point at all times. This situation can be considered in the same light as a potential hill with the top of the hill at the libration point and results in a destabilizing effect for all values of ξ and η, except for $\xi = \eta = 0$. The velocity-dependent terms in equations (7), on the other hand, contribute generally stabilizing accelerations for clockwise motion. This fact does not necessarily mean that a condition of overall stable motion requires clockwise motion with respect to the libration point at all times, since angular momentum is not conserved. The dynamics of the motion, however, as evidenced by equations (7), lead to predominantly clockwise motion.

Solution of the Equations of Motion

It is desired to find the solution of the equations of motion in the transformed coordinate system, as developed in the previous section:

$$\left. \begin{array}{l} \dfrac{d^2\xi}{dt^2} - 2\dfrac{d\eta}{dt} = \alpha\xi = 2.972795\xi \\[4mm] \dfrac{d^2\eta}{dt^2} + 2\dfrac{d\xi}{dt} = \beta\eta = 0.0272051\eta \end{array} \right\} \tag{10}$$

To solve these equations, let

$$\left. \begin{array}{l} \xi = K_n e^{\lambda_n' t} \\[3mm] \eta = L_n e^{\lambda_n' t} \end{array} \right\} \tag{11}$$

which results in

$$\left. \begin{array}{l} \dfrac{d\xi}{dt} = K_n \lambda_n' e^{\lambda_n' t} \\[3mm] \dfrac{d^2\xi}{dt^2} = K_n \lambda_n'^2 e^{\lambda_n' t} \\[3mm] \dfrac{d\eta}{dt} = L_n \lambda_n' e^{\lambda_n' t} \\[3mm] \dfrac{d^2\eta}{dt^2} = L_n \lambda_n'^2 e^{\lambda_n' t} \end{array} \right\} \tag{12}$$

Substituting the relations (11) and (12) into (10), dividing out the common factors, and rearranging gives

$$\left.\begin{aligned} \left(\lambda_n'^2 - \alpha\right)K_n - 2\lambda_n' L_n = 0 \\ 2\lambda_n' K_n + \left(\lambda_n'^2 - \beta\right)L_n = 0 \end{aligned}\right\} \tag{13}$$

For solutions of the form of equations (11), other than the already known solution $\xi = 0$, $\eta = 0$, the determinant of equations (13) must vanish. This condition can be expressed as

$$\begin{vmatrix} \lambda_n'^2 - \alpha & -2\lambda_n' \\[2ex] 2\lambda_n' & \lambda_n'^2 - \beta \end{vmatrix} \equiv \lambda_n'^4 - (\alpha + \beta - 4)\lambda_n'^2 + \alpha\beta = 0 \tag{14}$$

The solution of equation (14) is

$$\lambda_n'^2 = \frac{(\alpha + \beta - 4) \pm \sqrt{(\alpha + \beta - 4)^2 - 4\alpha\beta}}{2}$$

With the numerical values for α and β as given in equations (10), for mass ratio μ of $\frac{1}{82.45}$, the four solutions for λ_n' are

$$\left.\begin{aligned} \lambda_1' = -\lambda_2' = 0.297913\ i \\ \lambda_3' = -\lambda_4' = 0.954593\ i \end{aligned}\right\} \tag{15}$$

where $i = \sqrt{-1}$.

The general solution for the position and velocity components of the small body can now be written in the form

$$\left.\begin{aligned} \xi &= K_1 e^{\lambda_1' t} + K_2 e^{\lambda_2' t} + K_3 e^{\lambda_3' t} + K_4 e^{\lambda_4' t} \\[1ex] \eta &= L_1 e^{\lambda_1' t} + L_2 e^{\lambda_2' t} + L_3 e^{\lambda_3' t} + L_4 e^{\lambda_4' t} \\[1ex] \frac{d\xi}{dt} &= K_1 \lambda_1' e^{\lambda_1' t} + K_2 \lambda_2' e^{\lambda_2' t} + K_3 \lambda_3' e^{\lambda_3' t} + K_4 \lambda_4' e^{\lambda_4' t} \\[1ex] \frac{d\eta}{dt} &= L_1 \lambda_1' e^{\lambda_1' t} + L_2 \lambda_2' e^{\lambda_2' t} + L_3 \lambda_3' e^{\lambda_3' t} + L_4 \lambda_4' e^{\lambda_4' t} \end{aligned}\right\} \tag{16}$$

13

In equations (16) the coefficients K_n and L_n are related through equations (13); in particular

$$L_n = \frac{\lambda_n'^2 - \alpha}{2\lambda_n'} K_n = \frac{2\lambda_n'}{\beta - \lambda_n'^2} K_n \qquad (17)$$

With the numerical values of α and β as given in equations (10) and those for the values of λ_n' as given in equations (15), the numerical relations between the L_n and the K_n, obtained from equation (17), are

$$\left.\begin{aligned}
L_1 &= (5.138328\ i)K_1 = \gamma_1 i K_1 \\[2mm]
L_2 &= -(5.138328\ i)K_2 = -\gamma_1 i K_2 \\[2mm]
L_3 &= (2.034397\ i)K_3 = \gamma_3 i K_3 \\[2mm]
L_4 &= -(2.034397\ i)K_4 = -\gamma_3 i K_4
\end{aligned}\right\} \qquad (18)$$

In equations (16) the values of K_n are, in general, complex constants. Since the values of λ_n' are all pure imaginaries and are related as in equations (15), real values for the position and velocity components will be obtained only if K_1 and K_2, and K_3 and K_4 are complex conjugates. Let the coefficients K_n be expressed as

$$\left.\begin{aligned}
K_1 &= M_1 + iN_1 \\
K_2 &= M_2 + iN_2 \equiv M_1 - iN_1 \\
K_3 &= M_3 + iN_3 \\
K_4 &= M_4 + iN_4 \equiv M_3 - iN_3
\end{aligned}\right\} \qquad (19)$$

and let the exponentials in equations (16) be expressed in the form

$$\left.\begin{aligned}
e^{\lambda_n' t} &= \cos \lambda_n t + i \sin \lambda_n t \\
e^{-\lambda_n' t} &= \cos \lambda_n t - i \sin \lambda_n t
\end{aligned}\right\} \qquad (20)$$

Substituting the expressions (18), (19), and (20) into equations (16) yields

$$\xi = 2M_1 \cos \lambda_1 t - 2N_1 \sin \lambda_1 t + 2M_3 \cos \lambda_3 t - 2N_3 \sin \lambda_3 t$$

$$\eta = -2M_1\gamma_1 \sin \lambda_1 t - 2N_1\gamma_1 \cos \lambda_1 t - 2M_3\gamma_3 \sin \lambda_3 t - 2N_3\gamma_3 \cos \lambda_3 t$$

$$\frac{d\xi}{dt} = -2M_1\lambda_1 \sin \lambda_1 t - 2N_1\lambda_1 \cos \lambda_1 t - 2M_3\lambda_3 \sin \lambda_3 t - 2N_3\lambda_3 \cos \lambda_3 t \tag{21}$$

$$\frac{d\eta}{dt} = -2M_1\gamma_1\lambda_1 \cos \lambda_1 t + 2N_1\gamma_1\lambda_1 \sin \lambda_1 t - 2M_3\gamma_3\lambda_3 \cos \lambda_3 t + 2N_3\gamma_3\lambda_3 \sin \lambda_3 t$$

Equations (21) are a set of four simultaneous equations with the four unknown coefficients M_1, M_3, N_1, and N_3. These coefficients can be determined in terms of the initial position and velocity components (at $t = 0$). Designating the initial position and velocity components as ξ_0, η_0, $\left(\frac{d\xi}{dt}\right)_0$, and $\left(\frac{d\eta}{dt}\right)_0$, equations (21) become, at $t = 0$,

$$
\begin{aligned}
2M_1 \qquad\qquad +2M_3 \qquad\qquad &= \xi_0 \\
-2\gamma_1 N_1 \qquad\qquad -2\gamma_3 N_3 &= \eta_0 \\
-2\lambda_1 N_1 \qquad\qquad -2\lambda_3 N_3 &= \left(\frac{d\xi}{dt}\right)_0 \\
-2\gamma_1\lambda_1 M_1 \qquad\qquad -2\gamma_3\lambda_3 M_3 \qquad\qquad &= \left(\frac{d\eta}{dt}\right)_0
\end{aligned}
\tag{22}
$$

The form of equations (22) is such that they represent two sets of two simultaneous equations, and it is clear that the unknown coefficients M_n and N_n are each dependent on only one of the initial position components and one of the initial velocity components. Substituting the constants as given in equations (15) and (18) into equations (22) and solving for M_n and N_n in terms of the initial conditions gives

$$
\begin{aligned}
M_1 &= 2.361143\xi_0 + 1.215817\left(\frac{d\eta}{dt}\right)_0 \\
N_1 &= -0.111027\eta_0 + 0.236616\left(\frac{d\xi}{dt}\right)_0 \\
M_3 &= -1.861143\xi_0 - 1.215817\left(\frac{d\eta}{dt}\right)_0 \\
N_3 &= 0.034650\eta_0 - 0.597627\left(\frac{d\xi}{dt}\right)_0
\end{aligned}
\tag{23}
$$

Equations (21) are also rewritten by using the values of the constants as given in equations (15) and (18) to obtain the relations

$$
\left.
\begin{aligned}
\xi &= 2(M_1 \cos \lambda_1 t - N_1 \sin \lambda_1 t) + 2(M_3 \cos \lambda_3 t - N_3 \sin \lambda_3 t) \\[2mm]
\eta &= (-10.2766)(M_1 \sin \lambda_1 t + N_1 \cos \lambda_1 t) - (4.0688)(M_3 \sin \lambda_3 t + N_3 \cos \lambda_3 t) \\[2mm]
\frac{d\xi}{dt} &= (-0.5958)(M_1 \sin \lambda_1 t + N_1 \cos \lambda_1 t) - (1.9092)(M_3 \sin \lambda_3 t + N_3 \cos \lambda_3 t) \\[2mm]
\frac{d\eta}{dt} &= (-3.0616)(M_1 \cos \lambda_1 t - N_1 \sin \lambda_1 t) + (-3.8840)(M_3 \cos \lambda_3 t - N_3 \sin \lambda_3 t)
\end{aligned}
\right\} \quad (24)
$$

Equations (24) are explicit relations for the position and velocity components of a small body in motion relative to the stable libration points, with the dependence on the initial conditions given in equations (23), and are the first-order solution to the equations of motion. As indicated by the form of equations (24), the motion is doubly periodic in the frequencies $\lambda_1 = 0.297913$ and $\lambda_3 = 0.954593$, which correspond (for mean Earth-Moon distance) to periods of 91.71 and 28.62 days, respectively. The solutions given in equations (24) are applicable for both stable libration points L_4 and L_5 in the transformed coordinate systems illustrated in figure 3.

Equations and Properties of the Orbits

In the solutions for the position components of the motion as given in equations (24) (and also in eqs. (21)), each position component can be considered as the sum of the contributions of individual motions with the frequencies λ_1 and λ_3. The equations can therefore be manipulated in order to eliminate the time and obtain equations of the orbits of the individual motions in the two frequencies.

For the motion with frequency λ_1, the long-period motion, the applicable portions of equations (21) are

$$
\left.
\begin{aligned}
\xi &= 2M_1 \cos \lambda_1 t - 2N_1 \sin \lambda_1 t \\[2mm]
\eta &= -2\gamma_1 M_1 \sin \lambda_1 t - 2\gamma_1 N_1 \cos \lambda_1 t
\end{aligned}
\right\} \quad (25)
$$

If the first of equations (25) is multiplied by γ_1 and the result is squared and added to the square of the second of equations (25), equations (25) reduce to the form

$$
\gamma_1^2 \xi^2 + \eta^2 = 4\gamma_1^2 \left(M_1^2 + N_1^2 \right) \quad (26)
$$

16

This is the equation of the orbit for the long-period motion and is an ellipse in which the semiaxes are defined in terms of γ_1 and the initial conditions, expressed in the relations of equations (23). The eccentricity of the orbit, which is defined as

$$e = \sqrt{1 - \frac{(\text{Semiminor axis})^2}{(\text{Semimajor axis})^2}} = \sqrt{1 - \frac{1}{\gamma_1^2}} \qquad (27)$$

is independent of the initial conditions and dependent only on the constant γ_1.

Similar considerations lead to the properties of the motion with frequency λ_3, the short-period motion, with the equation of the orbit

$$\gamma_3^2 \xi^2 + \eta^2 = 4\gamma_3^2 \left(M_3^2 + N_3^2 \right) \qquad (28)$$

The properties of the component elliptical motions which comprise the resultant motion of a small body about the stable libration points, with the adopted value of $\mu = \dfrac{1}{82.45}$ and with the Moon at mean distance from the Earth, are listed in table I. The expression for the M_n and N_n are given in terms of the initial conditions in equations (23). The semiaxes are expressed in nondimensional units, as a fraction of the Earth-Moon distance.

TABLE I

PROPERTIES OF THE ELLIPTICAL COMPONENTS OF THE MOTION ABOUT THE STABLE

LIBRATION POINTS

[The M_n and N_n are defined in terms of the initial conditions in equations (23); the semiaxes are expressed as a fraction of the Earth-Moon distance (384,403 km).]

Motion	Period, days	Semiminor axis (along ξ-axis)	Semimajor axis (along η-axis)	Eccentricity
Long-period	91.71	$2\sqrt{M_1^2 + N_1^2}$	$10.2767\sqrt{M_1^2 + N_1^2}$	0.9809
Short-period	28.62	$2\sqrt{M_3^2 + N_3^2}$	$4.0688\sqrt{M_3^2 + N_3^2}$	.8709

An interesting result pertaining to the properties of the resultant motion, particularly with respect to the limits of the motion, is obtained by adding the relations for the semiaxes of the component ellipses presented in table I.

This results in

$$\left. \begin{array}{l} \xi_{max} = 2\sqrt{M_1{}^2 + N_1{}^2} + 2\sqrt{M_3{}^2 + N_3{}^2} \\[2ex] \eta_{max} = 10.2767\sqrt{M_1{}^2 + N_1{}^2} + 4.0688\sqrt{M_3{}^2 + N_3{}^2} \end{array} \right\} \qquad (29)$$

Equations (29) define the extremities of the motion of a small body in the vicinity of the stable libration points, subject to the initial conditions. The limiting boundary of the motion is not exactly an ellipse but is sufficiently close to an ellipse so that equations (29) can be considered as the semiaxes of an ellipse defining the maximum excursions as a function of the initial conditions.

Since the two frequencies of the motion λ_1 and λ_3 are not commensurate, the resultant motion is nonperiodic. Within the bounds of the motion, defined by the resultant of the ellipses with properties as given in table I, the orbit of the small body traces out a figure somewhat similar to a Lissajous curve.

The question arises as to whether it might be possible to choose the initial conditions such that the resultant motion reduces to one of the simple elliptical orbits with frequency λ_1 or λ_3. It is clear that, for this situation to exist, the motion pertaining to one of the frequencies must be eliminated. The solution given in equations (21) or (24) indicates that this result might be accomplished by choosing the initial conditions such that either $M_1 = N_1 = 0$ or $M_3 = N_3 = 0$.

Consider the situation for which $M_1 = N_1 = 0$. If the defining relations for M_1 and N_1 given in equations (23) are used, $M_1 = N_1 = 0$ when the following ratios exist between the initial values of the position and velocity components:

$$\left. \begin{array}{l} \dfrac{\xi_o}{\left(\dfrac{d\eta}{dt}\right)_o} = -\dfrac{1.215817}{2.361143} = -0.514927 \\[4ex] \dfrac{\eta_o}{\left(\dfrac{d\xi}{dt}\right)_o} = \dfrac{0.236616}{0.111027} = 2.131157 \end{array} \right\} \qquad (30)$$

These equations indicate that the characteristics of the initial conditions are such as to lead to clockwise motion relative to the libration points, which is the condition for stable motion; therefore, simple elliptical motion with either of the characteristic frequencies is possible with proper choice of the initial conditions. For motion with the frequency λ_3, equations (30) in combination with equations (23) yield the following values for the coefficients M_3 and N_3

$$M_3 = -0.257464\left(\frac{d\eta}{dt}\right)_0$$

$$N_3 = -0.523782\left(\frac{d\xi}{dt}\right)_0$$

(31)

Similar considerations for motion with the frequency λ_1, for which $M_3 = N_3 = 0$, lead to the relations (from eqs. (23))

$$\frac{\xi_0}{\left(\frac{d\eta}{dt}\right)_0} = -\frac{1.215817}{1.861143} = -0.653264$$

$$\frac{\eta_0}{\left(\frac{d\xi}{dt}\right)_0} = \frac{0.597627}{0.034650} = 17.247532$$

(32)

and

$$M_1 = -0.326632\left(\frac{d\eta}{dt}\right)_0$$

$$N_1 = -1.678326\left(\frac{d\xi}{dt}\right)_0$$

(33)

The sets of equations (30) and (31) and (32) and (33) thus define the initial conditions for two infinite sets of periodic orbits about the libration point. In these orbits the small body executes simple elliptical motion, with the center of the ellipse at the libration point, and with the frequency λ_3 or λ_1, respectively. The characteristics of these periodic motions are identical to those listed for the respective motions in table I.

NUMERICAL CALCULATIONS AND COMPARISONS

In the course of the analysis of the properties of motion about the stable libration points, a number of numerical calculations have been made. These calculations involve the numerical integration of the equations of motion of the restricted three-body problem (eqs. (1)) and have been performed to illustrate some of the properties of the motion and to serve as an indication of the applicability of the analytical results. The equations of motion have been programed for numerical integration on a high-speed digital computer (IBM 7090), using the Runge-Kutta integration procedure with variable time interval. The results of the numerical integration provide the time histories of the position and velocity components of the small body as a function of the initial conditions. Some additional remarks concerning the integration program are given in reference 5.

A few typical numerical calculations are presented in this section. The initial conditions and plots are in general presented in terms of nondimensional units. For reference to dimensional units, for the Moon at its mean distance from the Earth, the unit of distance is 384,403 kilometers and the unit of velocity is 1,024.5 meters per second.

Typical Orbits About the Stable Libration Points

Two typical examples of orbits about libration point L_4 are presented in figures 4 and 5, for which the examples are designated case I and case II, respectively. For both of these cases, the small body is initially located at point L_4; for case I it is projected along the X' axis with an initial velocity of 3.07 meters per second, and for case II along the Y' axis with an initial velocity of 30.7 meters per second. The time, measured in days from the initial time, is indicated for various positions in the orbits. The plots for the two cases cover a time period of about 6 months.

The two cases are somewhat similar in appearance as regards the characteristic loops, but the maximum excursions are considerably different, as expected from the differences in initial velocity, and amount to some 17,300 kilometers for case I and 155,000 kilometers for case II. The dependence of the maximum excursions on the initial conditions is discussed in some detail in a subsequent section entitled "Initial Condition Error Analysis," but for the present purposes the cases in figures 4 and 5 have been chosen as typical of cases for which the assumptions of the first-order theory generally apply (case I) or do not apply (case II). The loops in the orbits indicate the nonconservation of angular momentum for these cases.

Elliptical Motion Components and Limiting Envelope

As discussed in the section "Equations and Properties of the Orbits," the motion about the stable libration points is the resultant of motion in two ellipses with different, but known, periods. The properties of the individual elliptical motions are given in table I as a function of the initial conditions. For the resultant motion, the maximum excursions along the ξ- and η-axes, also as functions of the initial conditions, are given by the expressions of equations (29).

With the initial conditions of the numerical integration example plotted in figure 4 (case I), the component ellipses and the limiting envelope of the motion, as determined from the expressions of table I, are presented in figure 6. As indicated, the orbit is contained within the predicted limiting envelope for the duration of the numerical integration, which was carried out for a time period of about 6 months. For this case, and for a number of other cases with reasonably small values of the initial position and velocity components, there were no indications that the motion would exceed the limits of the envelope within an extended time period.
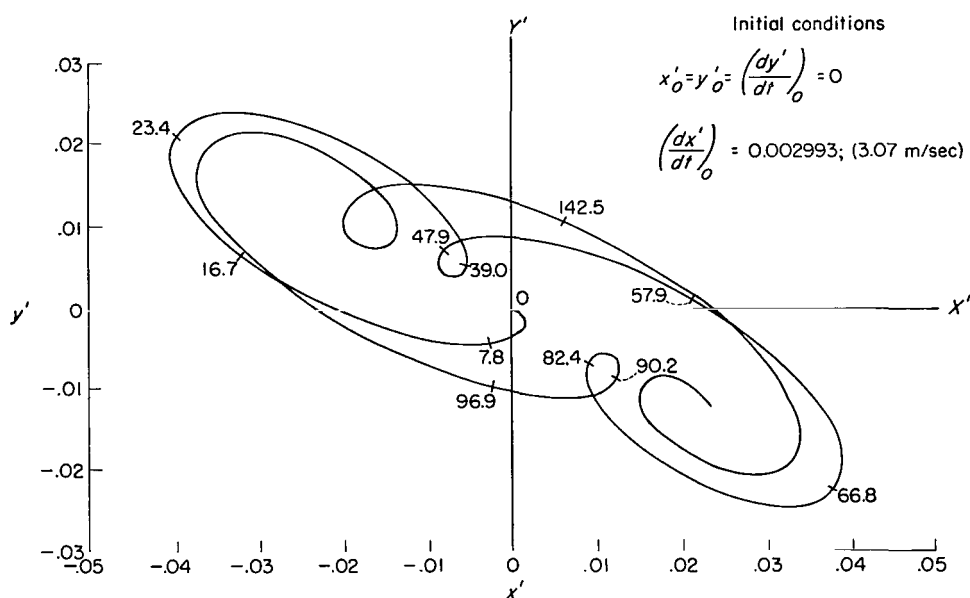
Figure 4.- Typical orbit about libration point L_4, case I. Numbers on curve denote time in days from the initial time.
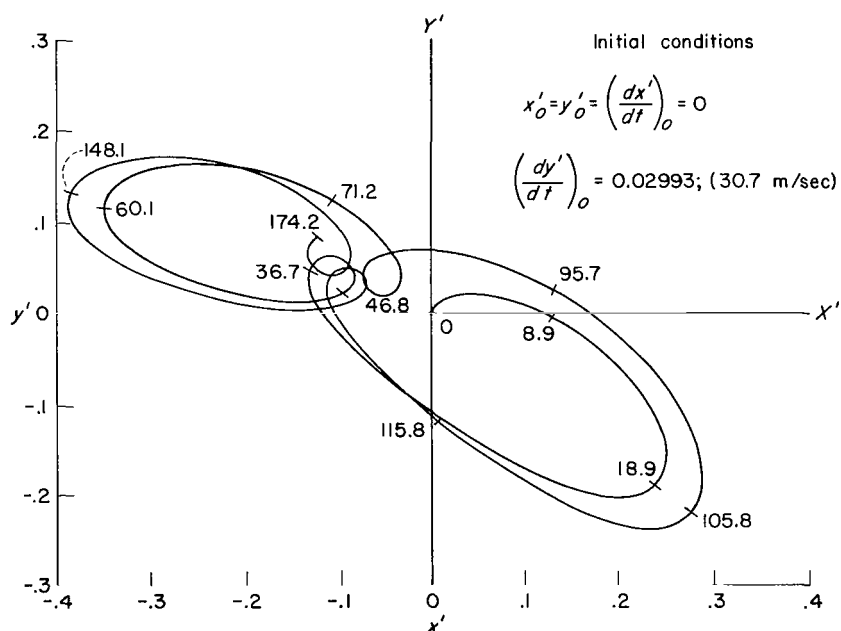

Figure 5.- Typical orbit about libration point L_4, case II. Numbers on curve denote time in days from the initial time.
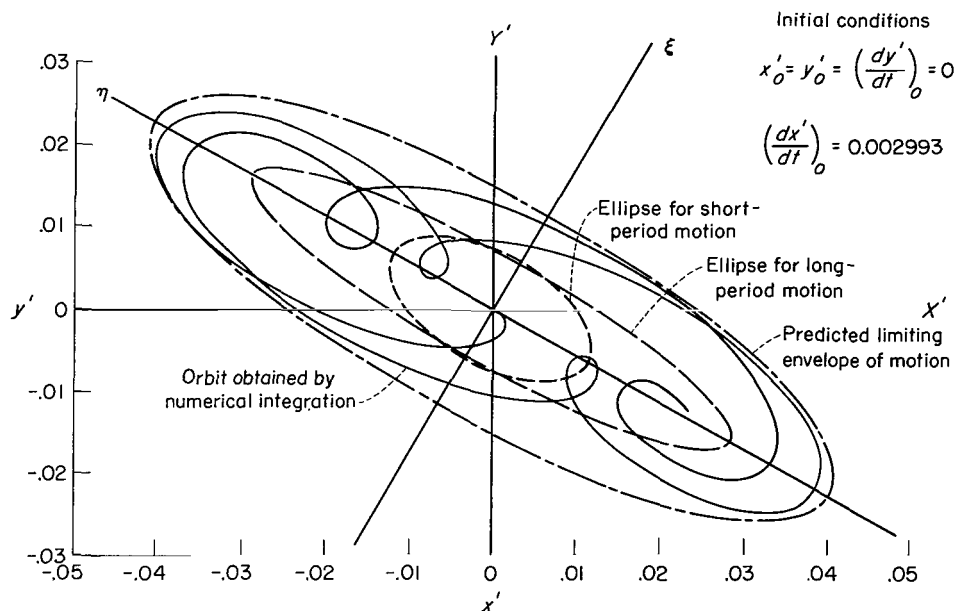
Figure 6.- Component ellipses, limiting envelope, and computed orbit for case I.

As an example of a case with initial conditions which result in an orbit which does exceed the limits of the predicted envelope, the predicted envelope and the computed orbit of case II are shown in figure 7. Such a situation is to be expected in view of the fact that an assumption of the analysis is that excursions from the libration point should be small compared with the Earth-Moon distance, whereas in this case the maximum excursions are of the order of 0.4 of this distance.

The First Integral and Solution of the Equations of Motion

A comparison of results predicted by equation (9) with a solution of the problem obtained by numerical integration is of interest. For this comparison, the numerical integration which resulted in the orbit plotted in figure 4 (case I) has been used. The initial conditions of case I and several positions in the orbit of case I were used to calculate the predicted velocity at these points by use of equation (9) and results are compared with the results as obtained in the numerical integration of case I. This comparison is presented in table II. The locations of particular points listed in table II are identified by the time notations in figure 4, or by reference to the position components. Table II indicates that the first integral predicts the velocity at any point within an accuracy of about 5 percent.
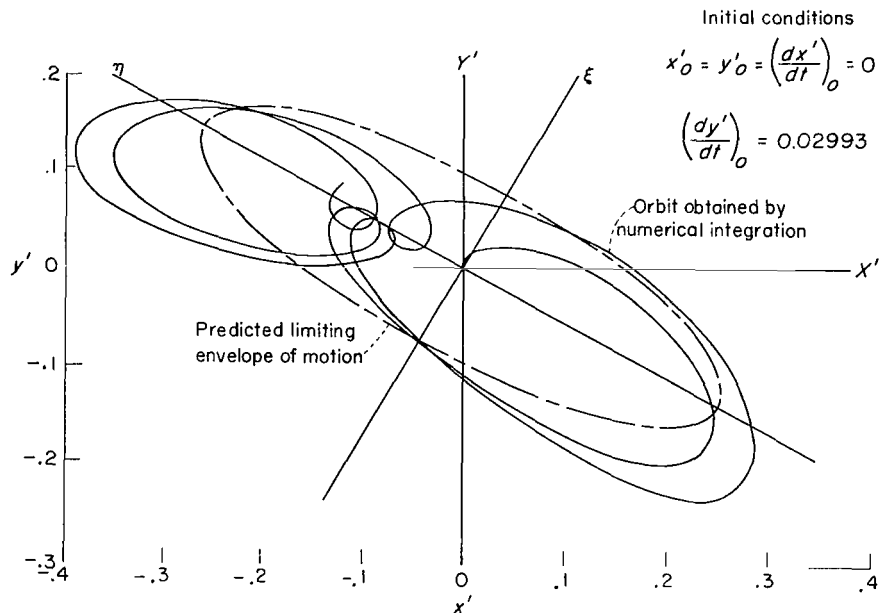
Figure 7.- Limiting envelope and computed orbit for case II.

TABLE II

COMPARISON OF RESULTS PREDICTED BY FIRST INTEGRAL WITH

RESULTS OF NUMERICAL INTEGRATION (CASE I)

Time, days	ξ	η	V, calculated from equation (9)		V, determined from numerical integration	
			Nondimensional	m/sec	Nondimensional	m/sec
16.70	-0.009958	0.031260	0.018175	18.60	0.017415	17.82
23.38	-.000418	.045404	.008097	8.28	.008001	8.19
57.89	.011470	-.018060	.020223	20.69	.020421	20.89
66.79	-.000633	-.043775	.007892	8.07	.007965	8.15
82.38	-.001413	-.010484	.004229	4.33	.004177	4.27
90.17	-.001100	-.015025	.004324	4.42	.004251	4.35
142.49	.012356	-.000171	.021514	22.01	.021435	21.93

The solutions for the position and velocity components as a function of the initial conditions are presented in equations (24). With the initial conditions of case I, it is of interest to compare the results of the position components as calculated from equations (24) with the results of the numerical integration as presented in figure 4. This comparison is shown in figure 8, where the predicted results from equations (24) are indicated by the small crosses and the corresponding results from the numerical integration, at the same value of time, are indicated by the small circles along the orbit. An examination of figure 8 indicates that equations (24) predict the orbital positions and properties of the orbit, including the loop, with acceptable accuracy throughout the time period.
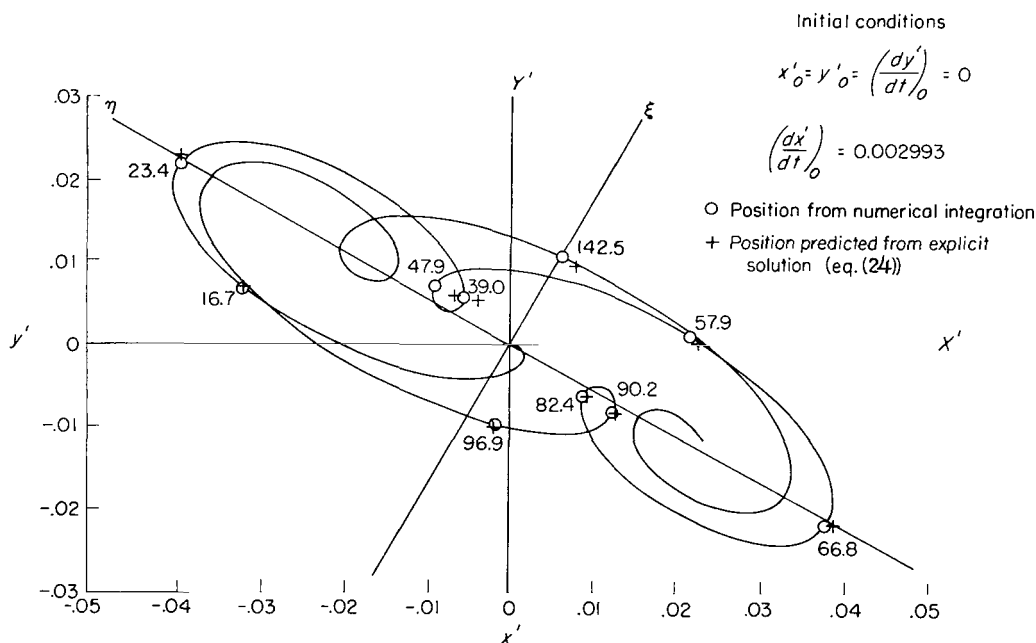

Figure 8.- Comparison of position in orbit obtained by explicit solution and by numerical integration; case I. Numbers on curve denote time in days from the initial time.

Periodic Orbits

The possibility of existence of periodic orbits about the libration points has previously been discussed. On the basis of the first-order analysis, it was concluded that it is possible to choose the initial conditions so as to obtain two infinite sets of periodic, elliptical orbits in the two characteristic frequencies. An investigation of the applicability of this conclusion of the first-order analysis is desirable.

With the initial conditions used for case I (fig. 4), the semiaxes of the predicted component elliptical motions were determined from the relations in table I. After the semiaxes were determined, equations (30) and (32) were used to calculate the value of the velocity component, to be applied at the corresponding

semiaxis position, which should lead to periodic elliptical motion in the particular characteristic frequency. For the present comparisons, the first of equations (30) and (32) were used to determine the values of $\left(\dfrac{d\eta}{dt}\right)_o$ to be applied at the ends of the semiminor axes, at positive values of ξ_o. When the necessary computations were completed, the initial conditions which should lead to periodic elliptical orbits, on the basis of the first-order analysis, were found to be:

Short-period motion (period of 28.62 days):

$$\left.\begin{array}{l} \xi_o = 0.0065660 \\[2mm] \left(\dfrac{d\eta}{dt}\right)_o = -0.0127513 \\[2mm] \eta_o = \left(\dfrac{d\xi}{dt}\right)_o = 0 \end{array}\right\} \tag{34}$$

Long-period motion (period of 91.71 days):

$$\left.\begin{array}{l} \xi_o = 0.00636101 \\[2mm] \left(\dfrac{d\eta}{dt}\right)_o = -0.00973728 \\[2mm] \eta_o = \left(\dfrac{d\xi}{dt}\right)_o = 0 \end{array}\right\} \tag{35}$$

Since these initial conditions are derived, through use of the first-order analysis, from the initial conditions of case I, the predicted periodic elliptical orbits are identically those which are presented as the component ellipses of figure 6. For comparison, the initial conditions as given in relations (34) and (35), after transformation to the coordinate system of the three-body problem, were inserted in the numerical integration program for the three-body problem. The comparison of the orbits so obtained from the numerical integration and the predicted periodic elliptical orbits are shown in figures 9 and 10. Figure 9 is for the long-period motion and figure 10 for the short-period motion.

The results shown in figures 9 and 10 indicate that the numerically integrated orbits obtained from integration of the three-body problem are not precisely elliptical orbits and are not exactly closed orbits. However, since the initial conditions for these orbits were obtained from a first-order analysis, the comparisons are considered to be reasonably good.

CONSIDERATIONS FOR ESTABLISHMENT OF ARTIFICIAL SATELLITES

NEAR THE STABLE LIBRATION POINTS

In view of the possibility that the stable libration points might be used as locations for establishment of artificial satellites, consideration of some of the pertinent factors involved is desirable. The problem is somewhat similar to that of establishing a satellite about the Moon. The departure of the satellite from the libration points as a function of residual errors after establishing the satellite, and considerations for minimization of this departure are the subjects of this section.

Initial Condition Error Analysis

Assume that a vehicle has been projected from Earth to the vicinity of either of the stable libration points, L_4 or L_5, and that a propulsion device has been used to reduce the velocity of the vehicle, relative to the libration point, to a reasonably small value. The accuracy with which this operation can be accomplished will depend on the guidance and control characteristics of the particular vehicle. The residual values of the distance and velocity components of the vehicle relative to the libration point, after application of the propulsive maneuver, are taken as initial conditions for determination of the maximum deviations of the vehicle with respect to the libration point.

Equations (29) give the expressions for the maximum excursion along the ξ- and η-axes as a function of the initial conditions, through the M_n and N_n as given in equations (23). Thus,

$$
\left.
\begin{aligned}
\xi_{max} = {}& 2\left\{\left[2.3611\xi_0 + 1.2158\left(\frac{d\eta}{dt}\right)_0\right]^2 + \left[-0.1110\eta_0 + 0.2366\left(\frac{d\xi}{dt}\right)_0\right]^2\right\}^{1/2} \\
& + 2\left\{\left[-1.8611\xi_0 - 1.2158\left(\frac{d\eta}{dt}\right)_0\right]^2 + \left[0.0347\eta_0 - 0.5976\left(\frac{d\xi}{dt}\right)_0\right]^2\right\}^{1/2} \\
\eta_{max} = {}& 10.2767\left\{\left[2.3611\xi_0 + 1.2158\left(\frac{d\eta}{dt}\right)_0\right]^2 + \left[-0.1110\eta_0 + 0.2366\left(\frac{d\xi}{dt}\right)_0\right]^2\right\}^{1/2} \\
& + 4.0688\left\{\left[-1.8611\xi_0 - 1.2158\left(\frac{d\eta}{dt}\right)_0\right]^2 + \left[0.0347\eta_0 - 0.5976\left(\frac{d\xi}{dt}\right)_0\right]^2\right\}^{1/2}
\end{aligned}
\right\} \quad (36)
$$

Equations (36) provide the means for performing an error analysis, giving the relations for the maximum excursions of the vehicle from the stable libration points as functions of any set of initial conditions. These equations can be used to determine the relative sensitivities of the maximum excursions to
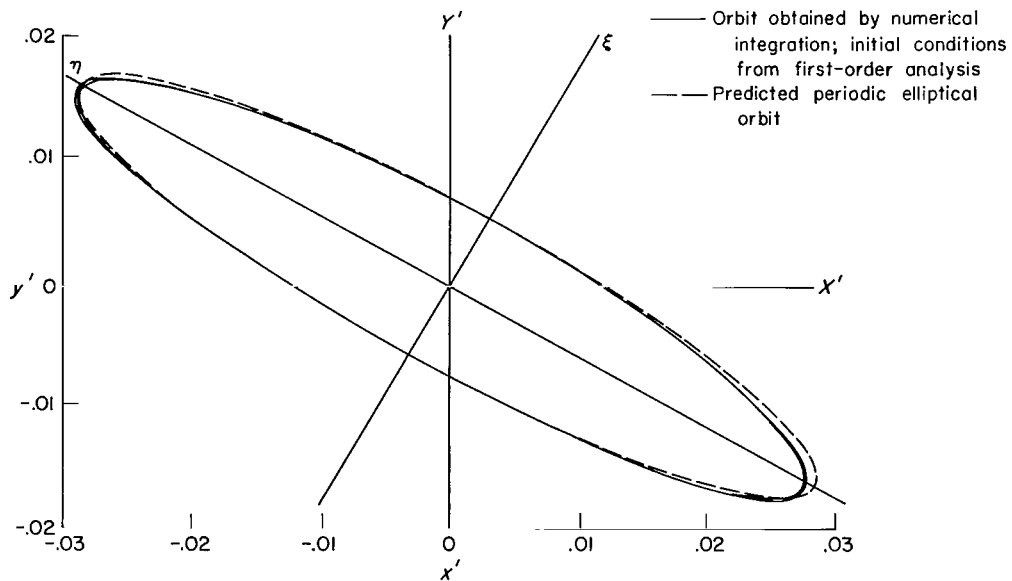
Figure 9.- Comparison of predicted periodic orbit and orbit obtained by numerical integration. Long-period motion, case I.
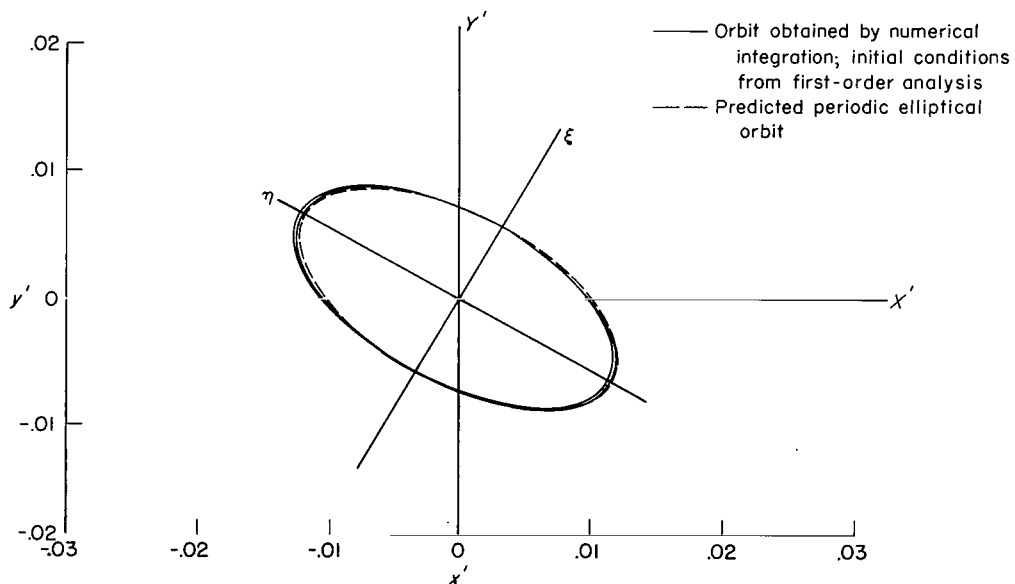

Figure 10.- Comparison of predicted periodic orbit and orbit obtained by numerical integration. Short-period motion, case I.

particular initial conditions and to indicate some general guidance and control requirements for establishing artificial satellites about the stable libration points.

Some calculated results utilizing equations (36) are presented in table III for combinations of initial position and velocity components taken in pairs. Combinations for which ξ_O and $\left(\frac{d\eta}{dt}\right)_O$ have opposite signs, and for which η_O and $\left(\frac{d\xi}{dt}\right)_O$ have the same sign, are those which result in initially clockwise motion with respect to the stable libration points.

TABLE III

INITIAL-CONDITION ERROR ANALYSIS

Initial conditions $\eta_O = \left(\frac{d\xi}{dt}\right)_O = 0$		Maximum excursion				Initial conditions $\xi_O = \left(\frac{d\eta}{dt}\right)_O = 0$		Maximum excursion			
		ξ_O and $\left(\frac{d\eta}{dt}\right)_O$ same sign		ξ_O and $\left(\frac{d\eta}{dt}\right)_O$ opposite sign				η_O and $\left(\frac{d\xi}{dt}\right)_O$ same sign		η_O and $\left(\frac{d\xi}{dt}\right)_O$ opposite sign	
ξ_O, km	$\left(\frac{d\eta}{dt}\right)_O$, m/sec	ξ_{max}, km	η_{max}, km	ξ_{max}, km	η_{max}, km	η_O, km	$\left(\frac{d\xi}{dt}\right)_O$, m/sec	ξ_{max}, km	η_{max}, km	ξ_{max}, km	η_{max}, km
0	0	0	0	0	0	0	0	0	0	0	0
	1	1,827	6,551	1,827	6,551		1	627	1,826	627	1,826
	3	5,482	19,659	5,482	19,659		3	1,880	5,482	1,880	5,482
	10	18,271	65,528	18,271	65,528		10	6,269	18,273	6,269	18,273
100	0	844	3,182	844	3,182	100	0	29	129	29	129
	1	2,671	9,734	983	3,369		1	597	1,698	656	1,955
	3	6,326	22,842	4,637	16,476		3	1,851	5,353	1,910	5,610
	10	19,116	68,710	17,427	62,346		10	6,240	18,144	6,298	18,401
500	0	4,219	15,907	4,219	15,907	500	0	145	639	145	639
	1	6,046	22,458	2,393	9,356		1	481	1,187	772	2,465
	3	9,701	35,566	1,262	3,752		3	1,735	4,842	2,026	6,121
	10	22,491	81,435	14,052	49,621		10	6,124	17,634	6,414	18,912
2,000	0	16,880	63,640	16,880	63,640	2,000	0	582	2,561	582	2,561
	1	18,707	70,191	15,053	57,088		1	577	2,001	1,209	4,387
	3	22,361	83,299	11,398	43,981		3	1,299	2,921	2,462	8,043
	10	35,151	129,167	1,999	5,009		10	5,687	15,712	6,851	20,834

Several pertinent results are indicated in table III. Residual velocity errors produce relatively greater maximum excursions than position errors; in terms of producing maximum deviations from the libration points, 1 meter per second in $\left(\frac{d\eta}{dt}\right)_O$ is equivalent to about 200 kilometers in ξ_O and 1 meter per second in $\left(\frac{d\xi}{dt}\right)_O$ is equivalent to about 1,450 kilometers in η_O. Errors in the

components ξ_O and $\left(\dfrac{d\eta}{dt}\right)_O$ produce considerably greater maximum deviations than

do errors of the same magnitude in the components η_O and $\left(\dfrac{d\xi}{dt}\right)_O$. Combinations

of initial conditions which result in initially clockwise motion relative to the libration points produce considerably smaller maximum deviations than those resulting in initially counterclockwise motion.

For comparison with the error-analysis results in the Earth-Moon plane given in table III, the excursions in the z'-direction due to initial conditions are easily calculated from equations (6). The maximum excursion in z', as obtained

from equations (6), is 385 kilometers for each meter per second of $\dfrac{dz'}{dt}$, plus the

initial displacement in z'. Thus, the sensitivity of the dispersion in z' to the initial position and velocity components in this direction is considerably less than the sensitivities of the in-plane excursions.

Minimization of the excursion distances of the vehicle from the libration points primarily involves minimization of the residual velocity components, par-

ticularly of the component $\left(\dfrac{d\eta}{dt}\right)_O$, after application of the propulsion device.

The dependence of the excursion distances on the initial direction of motion relative to the libration points suggests the possibility of use of bias in the nominal amount of incremental velocity to be applied to compensate partly for initial distance-component errors, assuming that such errors will be determinable through a tracking network while the vehicle is en route to the target.

Trajectory Considerations

Transfer trajectories for vehicles projected from the Earth to the libration points are very similar to those for vehicles projected from the Earth to the Moon. Except for the difference in the desired target point (libration point L_4 or L_5

instead of the Moon), the geometrical characteristics of the trajectories are almost identical. This similarity occurs because the gravitational attraction of the Moon has only a second-order effect on the geometrical characteristics of Earth-Moon trajectories. The studies reported in reference 5 are therefore directly applicable to considerations for establishing artificial satellites near the stable libration points.

Without prior knowledge of the guidance and control characteristics of a particular vehicle which might be used for establishing satellites at the stable libration points, it is somewhat difficult to discuss desirable trajectory design factors. A few general requirements, however, can be mentioned.

The accuracy requirements on the guidance system for placing the vehicle in the vicinity of the libration points are not particularly severe and are roughly comparable to those required to impact a vehicle somewhere on the surface of the Moon. On the other hand, the alinement of the thrust axis of the propulsive

device with the relative velocity vector, as well as control of the thrust or velocity increment to be applied, must be quite accurate.

As an illustration of these last points, reference 5 shows that the energy required to project a vehicle from the Earth to the distance of the Moon is slightly greater than 99 percent of the energy required to escape from the Earth. With this minimum energy, the relative approach velocity of the vehicle and the target libration point is about 900 meters per second. For a practical upper limit of trajectory energy, the escape energy, the relative approach velocity is about 1,400 meters per second. Thus 900 to 1,400 meters per second is the amount of velocity increment which must be eliminated by the propulsive device, preferably with an accuracy less than 1 percent. If the thrust axis is not precisely alined with the relative velocity vector, a velocity increment normal to the relative velocity vector of about 1.6 percent of the total velocity increment per degree of misalinement will result. This increment represents a relatively large residual velocity component (14 to 22 meters per second per degree misalinement) and could produce considerable maximum excursions of the vehicle from the libration point, as indicated by the numbers in table III.

In all probability, the vehicle guidance and control system will be better able to control the magnitude of the velocity increment than the particular inertial direction of the thrust axis required. In other words, better control and hence smaller residual velocity components will probably be obtained along the direction of the relative velocity vector than in the direction normal to the relative velocity vector. This has some bearing on the type of trajectory that might be used. From the results shown in reference 5, as applied to the present problem, the direction of the relative velocity vector as the vehicle approaches either L_4 or L_5 is more or less along the η-axis for low or nearly minimum energy trajectories, and along the ξ-axis for trajectories with the escape energy or higher. From the results of table III residual velocity increments along the η-axis are shown to produce greater excursions than those along the ξ-axis. Hence, it appears desirable to use low-energy trajectories, with the relative velocity vector along the η-axis, allowing the residual velocity increments due to misalinement of the propulsion thrust axis to lie in the direction of the ξ-axis (and the z-axis). This procedure has two additional advantages:

The deviations due to $\left(\dfrac{d\xi}{dt}\right)_o$ are not as sensitive to the direction of motion about the libration points as are those due to $\left(\dfrac{d\eta}{dt}\right)_o$, as shown in table III; and the low-energy trajectories do not require as large velocity increments as the higher-energy trajectories, thus reducing both the size of the propulsion device and the magnitude of the residual velocity increment due to a given amount of misalinement.

CONSIDERATIONS REGARDING NATURAL SATELLITES NEAR

THE STABLE LIBRATION POINTS

The existence of the Trojan group of asteroids, consisting of some 13 objects so far discovered in orbit about the stable libration points of the Sun-Jupiter system, naturally leads to speculation regarding the existence of objects in orbit about these points in the Earth-Moon system. The first report of objects in the vicinity of the stable libration points of the Earth-Moon systems is that of K. Kordylewski of the Cracow Observatory. (See refs. 3 and 4.) These objects are reported to be quite faint, a magnitude or two fainter than the gegenschein, and can be observed only under very favorable circumstances. The observations were an outcome of many years of searching in the vicinity of the libration points for single meteoroids, none of which as bright as magnitude 12 were found. Instead, two faintly luminous patches were reported, suggesting perhaps a swarm of tiny particles, none of which would be observable individually. The luminous patches are described as being at least 2° in diameter and separated by about 8°. There are very few additional details available regarding this report.

The purpose of this section is to consider several possible mechanisms for the accumulation of particles in the vicinity of the libration points and, although the observation reports are subject to further investigation, to attempt to provide some explanation of the data reported.

General Considerations

A considerable amount of attention is currently being directed to explanations of the concentration of dust near the Earth, an effect determined from recent rocket, satellite, and space-probe experiments. Some of this recent work is discussed in references 7, 8, and 9. These discussions are concerned with mechanisms which could lead to the dust concentration in the vicinity of the Earth, estimates of the concentration based on such mechanisms, and comparisons with the experimental results. It is of interest for the present discussion to consider some of these mechanisms in the different, but related, problem of accumulation of particles in the vicinity of the stable libration points.

A small proportion of the meteoroids which encounter the Earth-Moon system can be expected to graze the atmosphere of the Earth such that their energy is reduced sufficiently to be captured by the Earth. Of such captured particles, some should have sufficient energy to reach one of the stable libration points. However, from simple angular momentum considerations, it is easily shown that those which have only the minimum energy required to reach the libration points would still have velocities relative to the points of the order of 800 meters per second. On the basis of the present analysis of motion about the libration points, it would not be expected that trajectories of particles with a relative velocity of this magnitude would be noticeably affected in the vicinity of the libration points. Several numerical integration calculations have been made which confirm this expectation. Thus meteoroids which graze the Earth's atmosphere in all probability do not contribute to the particle population in the vicinity of the libration points.

In connection with the preceding discussion, Fremlin in reference 8 suggests that the gravitational field of the Moon could distort the orbits of those particles which have grazed the Earth's atmosphere and could cause some of the particles to have long-lived orbits about the Earth, thus contributing to the dust cloud about the Earth. It is physically possible, by this same process, for the gravitational field of the Moon to alter the angular momentum of these particles in such a manner that they could approach the libration points with considerably less relative velocity than the 800 meters per second mentioned above. This contingency would lead to the possibility of the trajectories being affected by the libration points. However, this process involves a series of low probability events. Only a small proportion of the particles encountering the Earth-Moon system will graze the atmosphere of the Earth in such a manner as to have the proper energy (a narrow band near the escape energy) to reach the distance of the Moon from the Earth. Of these, only a small proportion will pass sufficiently close to the Moon on the first pass to be affected by the gravitational field of the Moon; if they are not affected by the Moon on the first pass, they will impact the Earth on next approaching perigee. Finally, of those which are affected by the Moon, only a small proportion will be turned through the proper angle and will also remain sufficiently close to the Earth-Moon plane to reach one of the libration points. Such a process probably does not significantly contribute to the particle population in the vicinity of the libration points.

Mechanism for Particles of Lunar Origin

Whipple in reference 7 considers several possible explanations for the dust concentration near the Earth. Of the several possibilities considered, he discusses as most tenable a lunar origin for the dust, produced by expulsion of dust and droplets from the Moon by meteoritic crater formation. Within the range of energies of the particles ejected in lunar impact explosions, some of the particles would be injected into orbits about the Earth. The higher concentration of dust near the Earth would arise from convergence of the orbits and drag effects; thus the hypothesis of lunar origin of these particles is advanced as an explanation for the dust concentration near the Earth.

Whipple's hypothesis of a lunar origin for dust particles in orbits about the Earth can be directly applied to the present consideration of particle population near the stable libration points. Particles which approach the libration points will have their trajectories influenced by the dynamical properties of the points in proportion to their relative approach velocities. While there is no possibility of particles which approach the libration points from a large distance being captured in orbits about the points, except by collisions, there is the definite possibility that the trajectories of such particles can be significantly influenced near the points if the relative approach velocity is sufficiently small. Since the libration points have the same velocity relative to the Earth as does the Moon, a lunar origin for dust particles provides an interesting mechanism for a source of particles which can approach the libration points with small relative velocities. Particles which are ejected from the Moon with energies somewhat less than the escape energy from the Moon will, at a large distance from the Moon, have a small velocity relative to the Moon. If the direction of ejection of the particles is such as to direct them to the vicinity of the libration points, the

relative approach velocity to the libration points will thus also be small. The influence of the region of the libration point on the trajectories of the particles with small relative approach velocity can be such as to cause the particles to execute a partial orbit, or loop, near the libration point before proceeding away. As seen from the Earth, the temporary alteration of the trajectory in the vicinity of the libration points will result in an effective increase in density of particles in the vicinity of these points.

Some numerical calculations have been performed to illustrate the hypothesis discussed above. The equations of motion of the three-body problem have been numerically integrated to determine the trajectories of particles which leave the lunar surface with energies somewhat below the lunar escape energy and which proceed to the vicinity of the trailing libration point, L_5. Two typical trajectories are shown in figure 11. The trajectories originate at the trailing side of
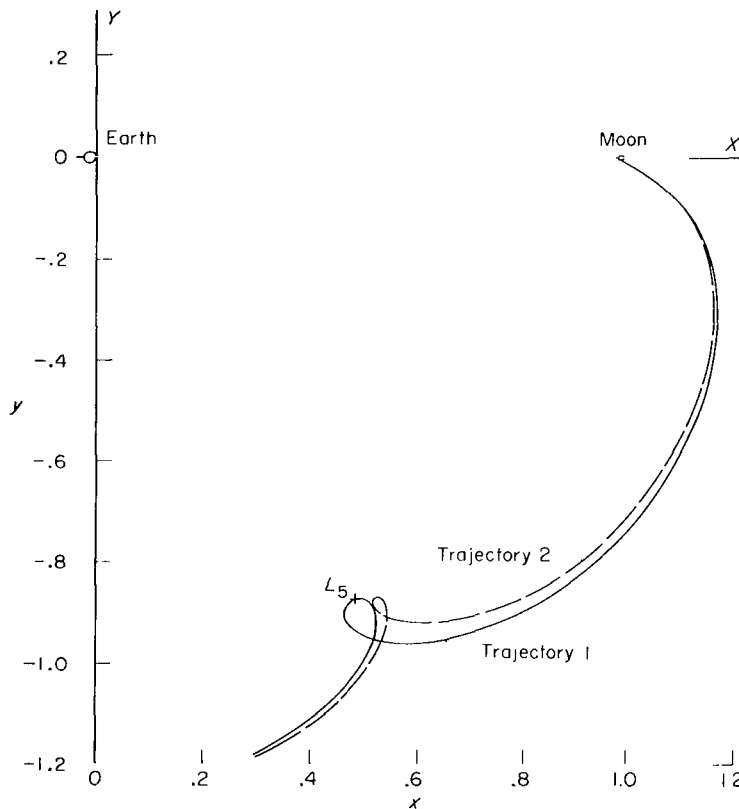

Figure 11.- Typical trajectories of particles from meteoritic impact on Moon to vicinity of point L_5.

the Moon and are injected with velocities relative to the Moon of 2,365.6 meters per second and 2,364.8 meters per second at angles with respect to the Earth-Moon line (measured away from the earth) of 38.0° and 37.5°, respectively, for the trajectories noted in figure 11 as 1 and 2. Trajectory 1 makes a loop about the libration point and trajectory 2 makes a loop in the vicinity of the point. Trajectories with other energies and injection directions make larger or smaller loops situated elsewhere in the vicinity of point L_5, depending on the initial

conditions. The transit times from the lunar surface to the point of closest approach to point L_5 are 22.7 and 20.5 days for trajectories 1 and 2, respectively.

If a steady stream of particles is assumed to leave the Moon along one of the trajectories in figure 11, the relative number of particles in any unit distance along the path of the trajectory will be a function of the velocity of the particles. In particular, as seen from the Earth, the relative density of particles in a unit geocentric angular segment will be inversely proportional to the geocentric angular velocity in that angular segment. Using the trajectories shown in figure 11, and by this procedure, the relative densities of particles along these trajectories, as viewed from the Earth, have been computed and are presented in figure 12. The relative densities have been normalized with respect to the density of particles in a segment close to the lunar surface.

The relative density plots shown in figure 12 indicate that a lunar origin for dust particles can result in an accumulation of transient particles in the vicinity of the libration points. Although this mechanism implies a continuous supply of particles, it should be mentioned that it is possible for particles which have once passed near the libration points to return repeatedly over sufficiently long time periods. Also, particles injected into Earth orbits with proper energy levels, and which do not approach the libration points on a direct trajectory, can at a later time pass in the vicinity and contribute to the particle density.

The fact that particles with proper energy levels make loops about or near the libration points, causing an increase in particle density and causing randomly oriented relative velocity vectors for the particles, also increases the probability of collisions resulting in captured orbits about the points. Such captured particles which eventually escape due to loosely captured orbits or disturbing perturbations are replaced by additional captured particles through this same process. It is possible that the captured particles provide a threshold of particle density required for observation and that the transient particle density supplements this.

Particles in Captured Orbits

Of the particles in captured orbits about the libration points, the distribution of energy between the two elliptical components of the motion determines the relative density of particles as observed from the Earth. As an example, consider the effect of a number of particles of which the orbits are ellipses, such that for these particles all the energy is distributed in a single component of the motion. As observed from the Earth, the line of sight being approximately along the ξ-axis as in figures 9 and 10, the density of particles would be relatively higher on either side of the libration point than at the libration point itself. The same situation would result as long as the distribution of energy between the two elliptical components of the motion was such that the semimajor axes of the ellipses were not nearly equal. Although it would be difficult to make estimates of the energy distribution, it seems probable that the energy would be predominantly distributed unequally between the two components of the
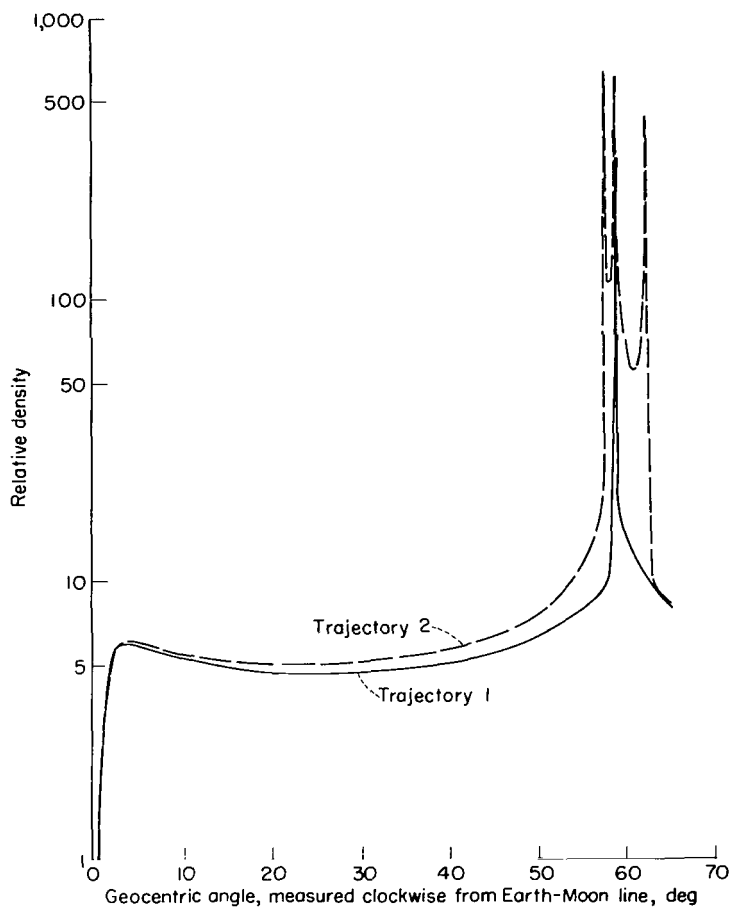
Figure 12.- Relative density of particles near point L_5, as seen from the Earth.

motion and, hence, that the particle density as observed from the Earth would be greater in regions on either side of the libration point than at the point. This assumption provides a qualitative explanation for the existence of two clouds of particles, as reported in references 3 and 4.

Suggestions for Future Observations

If a considerable proportion of the dust in the Earth-Moon system is of lunar origin, and if some of these particles follow more or less direct trajectories to the vicinity of the libration points, as discussed previously, it might be expected that the density of particles near the libration points would be increased during periods of concentrated meteoric activity. Relatively large numbers of meteoritic impacts on the Moon should occur at the same times that meteor showers are experienced at the earth. The most dependable and conspicuous annual meteor showers are the Perseids with maximum display on August 12, the Orionids with maximum display on October 20, and the Geminids display on December 10.

35

It was mentioned in connection with figure 11, however, that the transit times of particles on the illustrated trajectories, from the lunar surface to the libration points, were more than 20 days. It is therefore suggested that future observations or searches for clouds of particles near the libration points be intensified a few weeks after heavy meteor showers, assuming that other conditions are favorable. An increase in visual magnitude of the clouds at such times would give some confirmation of the hypotheses presented for the particle populations near the libration points and at the same time would give credence to the concept of a lunar origin for the dust concentration near the Earth. It might also be possible, on the basis of such observations, to arrive at some estimates of the amount of material ejected from the Moon by meteoritic impact.

CONCLUDING REMARKS

A first-order analysis of the equations of motion of the restricted three-body problem leads to an explicit solution for the position and velocity components of the motion about the stable libration points of the Earth-Moon system, and hence to determination of the properties of the motion as a function of the initial conditions. Comparisons of analytical results and numerical integration results are found to be in good agreement, even for rather large excursions from the libration points. On the basis of the analysis, consideration is given to factors involved in establishment of artificial satellites about the stable libration points, and suggestions are given for possible scientific experiments which might be performed with use of such satellites. The suggestions made for future observations of the clouds of particles could lead to some interesting new information regarding the origin of particles in the Earth-Moon system.

Langley Research Center,
    National Aeronautics and Space Administration,
        Langley Station, Hampton, Va., October 4, 1962.

REFERENCES

1. Moulton, Forest Ray: An Introduction to Celestial Mechanics. Second rev. ed.,
   The Macmillan Co., c.1914.

2. Plummer, H. C.: An Introductory Treatise on Dynamical Astronomy. Dover Pub.,
   Inc., 1960.

3. Anon.: New Natural Satellites of the Earth. Sky and Telescope (News Notes),
   vol. XXII, no. 1, July 1961, p. 10.

4. Anon.: More About the Earth's Cloud Satellites. Sky and Telescope, vol. XXII,
   no. 2, Aug. 1961, pp. 63 and 83.

5. Michael, William H., Jr., and Tolson, Robert H.: Three-Dimensional Lunar
   Mission Studies. NASA MEMO 6-29-59L, 1959.

6. Benedikt, E. T.: Exact Determination of the Lunar Mass by Means of Selenoid
   Satellites. Nature (Letters to the Editors), vol. 192, no. 4801, Nov. 4,
   1961, pp. 442-443.

7. Whipple, Fred L.: The Dust Cloud About the Earth. Nature, vol. 189, no. 4759,
   Jan. 14, 1961, pp. 127-128.

8. Fremlin, J. H., Beard, David B., and Whipple, Fred L.: The Dust Cloud About
   the Earth. Nature, vol. 191, no. 4783, July 1, 1961, pp. 31-34.

9. Singer, S. F.: Interplanetary Dust Near the Earth. Nature, vol. 192,
   no. 4800, Oct. 28, 1961, pp. 321-323.